CONSTRUCTION OF GREEN'S FUNCTION IN TERMS OF GREEN'S FUNCTION OF LOWER DIMENSION

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A method of constructing a Green's tensor for systems of linear differential equations with constant coefficients, defined in a space of odd dimensionality, in terms of the Green's tensor for a hyperplane, is given. Fundamental problems of the classical theory of elasticity of the internal stress theory and of the dislocation theory, are used as examples of the application of the derived method to the problems in the field theory in anisotropic media.

1. It is well known, that an n-dimensional scalar, or tensor Green's function $G(x)$ can be reduced, using the method of descent, to a Green's function $\Phi(\mathbf{x}, \boldsymbol{\tau})$ for an arbitrary (n-1)-dimensional subspace $x_i \tau_i = 0$, orthogonal to the vector $\boldsymbol{\tau}$. In an unbounded space we have

$$\Phi(\mathbf{x}, \boldsymbol{\tau}) = \int_{-\infty}^{\infty} G(\mathbf{x} - \boldsymbol{\tau} s)\, \tau ds \tag{1.1}$$

where τ is the length of the vector $\boldsymbol{\tau}$. Since $\Phi(\mathbf{x}, \boldsymbol{\tau})$ is a function of the first argument, therefore, it is not a function of the n-dimensional vector $\mathbf{x}$, it is a function of the $(n-1)$-dimensional vector $\mathbf{x} - \boldsymbol{\tau}(\mathbf{x}\,\boldsymbol{\tau})\,\tau^{-2}$.

We shall show that, for a wide class of equations given in an infinite, odd-dimensional space, solution of the converse problem is possible.

2. L e m m a. Suppose that a function is given in an n-dimensional space, satisfying the condition

$$G(\alpha \mathbf{x}) = \alpha^{-k} \operatorname{sign} \alpha\, G(\mathbf{x}) \tag{2.1}$$

where k is a natural number. Then $G(\mathbf{x})$ can be determined in terms of the function $\Phi(\mathbf{x}, \boldsymbol{\tau})$ defined by (1.1), with the help of the following expressions:

$$G(\boldsymbol{\tau}) = -\frac{1}{2\tau} x_i \frac{\partial}{\partial x_i} \Phi(\mathbf{x}, \boldsymbol{\tau}) \qquad (k = 1) \tag{2.2}$$

$$G(\boldsymbol{\tau}) = \frac{(-1)^{k-1}}{2(k-2)!} D^{k-1} \frac{\Phi(\mathbf{x}, \boldsymbol{\tau})}{\tau} \qquad (k \geqslant 2) \qquad\qquad D = x_j \partial(\ldots) / \partial \tau_j \tag{2.3}$$

P r o o f. 2.1°. If $k = 1$, the integral (1.1) diverges logarithmically and the definition of $\Phi(\mathbf{x}, \boldsymbol{\tau})$ must therefore be made more precise. Nevertheless, the gradient of this function can be determined uniquely

$$\frac{\partial}{\partial x_i} \Phi(\mathbf{x}, \boldsymbol{\tau}) = \Phi_{,i}(\mathbf{x}, \boldsymbol{\tau}) = \int_{-\infty}^{\infty} G_{,i}(\mathbf{x} - \boldsymbol{\tau} s)\, \tau\, ds \tag{2.4}$$

Let us multiply (2.4) by x_i. Using the Euler's theorem for homogeneous functions, we obtain

$$x_i\Phi_{,i}(\mathbf{x}, \boldsymbol{\tau}) = -\int_{-\infty}^{\infty} G(\mathbf{x}-\boldsymbol{\tau}s)\,\tau\,ds + \int_{-\infty}^{\infty} s\tau_i G_{,i}(\mathbf{x}-\boldsymbol{\tau}s)\,\tau\,ds$$

Observing that

$$\tau_i G_{,i}(\mathbf{x}-\boldsymbol{\tau}s) = -\frac{d}{ds}G(\mathbf{x}-\boldsymbol{\tau}s)$$

and integrating its right-hand side by parts, we obtain

$$x_i\Phi_{,i}(\mathbf{x}, \boldsymbol{\tau}) = -s\tau G(\mathbf{x}-\boldsymbol{\tau}s)\Big|_{s=-\infty}^{s=\infty} = -2\tau G(\boldsymbol{\tau})$$

2.2°. When $k > 2$, we apply the operator D to both parts of the Eq.

$$\frac{\Phi(\mathbf{x}, \boldsymbol{\tau})}{\tau} = \int_{-\infty}^{\infty} G(\mathbf{x}-\boldsymbol{\tau}s)\,ds \tag{2.5}$$

and use the Euler's theorem to obtain

$$D\frac{\Phi(\mathbf{x}, \boldsymbol{\tau})}{\tau} = -\int_{-\infty}^{\infty} sx_i G_{,i}(\mathbf{x}-\boldsymbol{\tau}s)\,ds = k\int_{-\infty}^{\infty} sG(\mathbf{x}-\boldsymbol{\tau}s)\,ds - \int_{-\infty}^{\infty} s^2\tau_i G_{,i}(\mathbf{x}-\boldsymbol{\tau}s)\,ds$$

This on integration by parts, yields

$$D\frac{\Phi(\mathbf{x}, \boldsymbol{\tau})}{\tau} = (k-2)\int_{-\infty}^{\infty} sG(\mathbf{x}-\boldsymbol{\tau}s)\,ds + s^2G(\mathbf{x}-\boldsymbol{\tau}s)\Big|_{s=-\infty}^{s=\infty}$$

hence

$$D\frac{\Phi(\mathbf{x}, \boldsymbol{\tau})}{\tau} = -2G(\boldsymbol{\tau}) \quad (k=2); \quad D\frac{\Phi(x, \tau)}{\tau} = \int_{-\infty}^{\infty} (k-2)\,sG(\mathbf{x}-\boldsymbol{\tau}s)\,ds \quad (k>2)$$

Applying the operator D^{k-2} to both parts of (2.5), we obtain

$$D^{k-2}\frac{\Phi(\mathbf{x}, \boldsymbol{\tau})}{\tau} = (k-2)!\int_{-\infty}^{\infty} s^{k-2}G(\mathbf{x}-\boldsymbol{\tau}s)\,ds$$

Then

$$D^{k-1}\frac{\Phi(\mathbf{x}, \boldsymbol{\tau})}{\tau} = (k-2)!\,s^kG(\mathbf{x}-\boldsymbol{\tau}s)\Big|_{s=-\infty}^{s=\infty}$$

from which, taking into account (2.1), we obtain (2.4).

3. Let us obtain the conditions of applicability of the Lemma of Section 2 to the Green's tensors. To shorten the notation, we shall employ Greek letters for the indices assuming n^l values, and use a single index to denote the components of tensors of rank l. (As before, Latin alphabet will be used for the indices assuming n values).

Let a linear, homogeneous, m-th order differential operator with constant coefficients

$$L_{\alpha\beta} = A_{\alpha\beta p_1\ldots p_m}\nabla_{p_1}\ldots\nabla_{p_m}$$

be given in an n-dimensional unbounded space.

We shall consider the equation

$$L_{\alpha\beta}\psi_\beta(\mathbf{x}) + f_\alpha(\mathbf{x}) = 0 \tag{3.1}$$

and prove the following theorem.

T h e o r e m 1. A Green's tensor of the system (3.1) or its derivatives, can be constructed in odd-dimensional spaces in terms of a Green's tensor for hyperplanes, using relations of the type (2.2) and (2.3).

P r o o f. We shall assume that a Green's tensor $G_{\beta\gamma}(\mathbf{x})$ exists, satisfying the Eq.

$$L_{\alpha\beta}G_{\beta\gamma}(\mathbf{x}) + \delta_{\alpha\gamma}\delta(\mathbf{x}) = 0 \tag{3.2}$$

Taking into account the fact that

$$G_{\beta\gamma}(\alpha\mathbf{x}) = \alpha^{m-n}(\operatorname{sign}\alpha)^m G_{\beta\gamma}(\mathbf{x})$$

and putting $n - m = k$, we have

$$G_{\beta\gamma}(\alpha\mathbf{x}) = \begin{cases} \alpha^{-k} G_{\beta\gamma}(\mathbf{x}), & \text{if } n \text{ is even} \quad (3.3) \\ \alpha^{-k} \operatorname{sign}\alpha G_{\beta\gamma}(\mathbf{x}) & \text{if } n \text{ is odd} \quad (3.4) \end{cases}$$

If $k < 0$, then we consider the derivatives of the Green's tensor

$$G_{\beta\gamma,\, p_1,\ldots,p_{1-k}}(\alpha\mathbf{x}) = \begin{cases} \alpha^{-1} G_{\beta\gamma,\, p_1,\ldots,p_{1-k}}(\mathbf{x}), & \text{if } n \text{ is even} \quad (3.5) \\ \alpha^{-1} \operatorname{sing}\alpha G_{\beta\gamma,\, p_1,\ldots,p_{1-k}}(\mathbf{x}), & \text{if } n \text{ is odd} \quad (3.6) \end{cases}$$

Solution of the problem for the hyperplane $x_i\tau_i = 0$ can be obtained with help of the Green's tensor $\Phi_{\beta\gamma}(\mathbf{x}, \tau)$ defined by the following system of equations

$$L_{\alpha\beta}\Phi_{\beta\gamma}(\mathbf{x}, \tau) + \delta_{\alpha\gamma}\int_{-\infty}^{\infty} \delta(\mathbf{x} - \tau s)\,\tau ds \tag{3.7}$$

If $k > 1$, we have

$$\Phi_{\beta\gamma}(\mathbf{x}, \tau) = \int_{-\infty}^{\infty} G_{\beta\gamma}(\mathbf{x} - \tau s)\,\tau\, ds \tag{3.8}$$

If $k < 0$, we can obtain the derivatives of the Green's tensor

$$\Phi_{\alpha\beta,\, p_1,\ldots,p_{1-k}}(\mathbf{x}, \tau) = \int_{-\infty}^{\infty} G_{\alpha\beta,\, p_1,\ldots,p_{1-k}}(\mathbf{x} - \tau s)\,\tau\, ds \tag{3.9}$$

Combining the relations (3.4), (3.6), (3.8) and (3.9) with the conditions of the Lemma, we confirm the validity of the theorem.

4. Let us now assume that the field ψ cannot be constructed with the help of the point sources, i.e. that the system (3.1) does not admit the construction of the ordinary Green's tensor (3.2) and, that only a generalized Green's tensor $G_{\alpha q_1\ldots q_r}(\mathbf{x})$ exists, corresponding to an elementary extended source of ψ (see e.g. [1]). We find, that for the subspace $x_k\tau_k = 0$, the generalized Green's tensor $\Phi_\alpha(\mathbf{x}, \tau)$ has a projection on the τ-direction and is, therefore, of lower rank than the tensor $G_{\alpha q_1\ldots q_r}(\mathbf{x})$

$$\Phi(\mathbf{x}, \tau) = \tau_{q_1}\ldots\tau_{q_r}\tau^{1-r}\int_{-\infty}^{\infty} G_{q_1\ldots q_r}(\mathbf{x} - \tau s)\, ds \tag{4.1}$$

(indices not involved in the contraction are omitted). We find that the Lemma of Section 2 is not valid for (4.1), the following theorem can, however, be formulated.

T h e o r e m 2. If the generalized Green's tensor $G_{\alpha q_1\ldots q_r}$ related to the generalized Green's tensors for the hyperplane $x_i\tau_i = 0$ by (4.1) can be represented in the form

$$G_{\alpha q_1\ldots q_r} = e_{q_1p_1}^{t_1h_1}\ldots e_{q_rp_r}^{t_rh_r} x_{p_1}\ldots x_{p_r} G'_{\alpha t_1\ldots t_r h_1\ldots h_r} \tag{4.2}$$

where $e_{pq}^{ij} = \delta_p^i\delta_q^j - \delta_q^i\delta_p^j$, and $\mathbf{G}'$ satisfies (2.1), then

$$G_{\alpha q_1\ldots q_r}(\tau) = \frac{(-1)^{k+r-1}}{2r!\,(k-2)!}\frac{\partial^r}{\partial x_{q_1}\ldots\partial x_{q_r}} D^{k-1}\frac{\Phi_\alpha(\mathbf{x}, \tau)}{\tau^{1-r}} \tag{4.3}$$

P r o o f. Let us insert (4.3) into (4.2) and apply to both sides the operator D^{k-1}. Any function of the bivector

$$y^{ij} \equiv e_{pq}^{ij}\tau_p x_q$$

will become zero under the action of this operator, therefore

$$D^{k-1}\frac{\Phi_\alpha(\mathbf{x},\tau)}{\tau^{1-r}}=y^{p_1q_1}\dots y^{p_rq_r}D^{k-1}\int_{-\infty}^{\infty}G'_{\alpha p_1\dots p_rq_1\dots q_r}(\mathbf{x}-\tau s)\,ds$$

Applying the results of the Lemma to the latter expression, we obtain

$$\frac{(-1)^{k-1}}{2(k-2)!}D^{k-1}\frac{\Phi_\alpha(\mathbf{x},\tau)}{\tau^{1-r}}=y^{p_1q_1}\dots y^{p_rq_r}G'_{\alpha p_1\dots p_rq_1\dots q_r}(\tau) \tag{4.4}$$

The r-fold contraction of the tensor $\mathbf{G}'$ with the bivector y^{ij} in the right-hand side of (4.4), will be an r-th degree polynomial in the vector $\mathbf{x}$, hence applying the operator $\partial^r(\dots)/\partial x_{q_1}\dots\partial x_{q_r}$ to both parts of (4.4), we obtain the required formula (4.3).

5. As an example, we shall consider some typical three-dimensional problems of the theory of elasticity of anisotropic media, reducing them to investigation of plane problems, in which the complex variable methods can be utilized. Various types of sources appearing in the classical theory of elasticity (potential fields, ordinary Green's functions), in the dislocation theory (vortex fields, generalized Green's functions) and in the internal stress theory (potential and bivortical fields and corresponding ordinary and generalized Green's functions depending on the definition of the source) can furnish various examples of application of the above theorems. Classification of the sources, the terminology and the basic equations, are taken from [2 and 3].

5.1°. Green's tensor for a concentrated force. The Green's tensor G_{ij} satisfying Eq.

$$C_{ijkl}G_{km,lj}(\mathbf{x})+\delta_{im}\delta(\mathbf{x})=0 \tag{5.1}$$

defines the displacement field u_i for a unit concentrated force, in an anisotropic medium with elastic moduli C_{ijkl}.

We know [4] that the solution of (5.1) can be obtained in the explicit form only for some isolated cases. In the two-dimensional case of the plane $x_i\tau_i=0$, the displacement field is described by the Green's tensor $\Phi_{ij}(\mathbf{x},\tau)$ satisfying the equation

$$C_{ijkl}\Phi_{km,lj}(\mathbf{x},\tau)+\delta_{im}\int_{-\infty}^{\infty}\delta(\mathbf{x}-\tau s)\,\tau ds$$

By the Theorem 1 we have

$$G_{ij}(\tau)=-\frac{1}{2\tau}x_k\frac{\partial}{\partial x_k}\Phi_{ij}(\mathbf{x},\tau) \tag{5.2}$$

5.2°. Green's tensor for internal deformations. If the stresses are caused not by external forces but by internal deformations ε°_{ij} of arbitrary origin (e.g. thermoelastic, strictional, plastic e.a.), then the displacement field satisfies

$$C_{ijkl}u_{k,lj}=-C_{ijkl}\varepsilon^\circ_{kl,j} \tag{5.3}$$

to which corresponds the following Green's tensor:

$$G_{ijk}=\sigma^k_{ij}=C_{ijlm}G_{lk,m} \tag{5.4}$$

Here $\sigma_{ij}{}^k$ denote the ij-component of the field of stress caused by a unit force acting [2, and 5] in the k-direction, while G_{lk} is the Green's tensor for the concentrated force.

In the two-dimensional case we have

$$\Phi_{ijk}=C_{ijlm}\Phi_{lk,m} \tag{5.5}$$

By Theorem 1,

$$\sigma^k_{ij}(\tau)=-\frac{1}{2}x_l\frac{\partial}{\partial\tau_l}\frac{\Phi_{ijk}(\mathbf{x},\tau)}{\tau} \tag{5.6}$$

i.e.

$$u_k(\mathbf{x}) = -\frac{1}{2}\tau_l \frac{\partial}{\partial x_l}\int \Phi_{ijk}(\tau, \mathrm{X})\, \varepsilon_{ij}{}^\circ(\mathbf{x}')\frac{(d\mathbf{x}')}{R} \quad (\mathrm{X} = \mathbf{x} - \mathbf{x}',\ R = |\mathrm{X}|)$$

For elastic deformations (5.7)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \varepsilon_{ij}{}^\circ = \int \left\{\frac{1}{2}[\sigma^i_{mn,j}(\mathrm{X}) + \sigma^j_{mn,i}(\mathrm{X})] - \delta_{in}\delta_{jm}\delta(\mathrm{X})\right\}\overset{\circ}{\varepsilon}_{mn}(\mathbf{x}')\,(d\mathbf{x}')$$

taking into account the fact that the index of homogeneity of the functions $\sigma^k_{ij,l}(\mathbf{x})$ and $\delta(\mathbf{x})$ is equal to -3 we have according to Theorem 1, the following relation:

$$\varepsilon_{ik}(\mathbf{x}) = \frac{1}{2}\tau_m\tau_n\frac{\partial^2}{\partial x_m \partial x_n}\int E^{ik}_{jl}(\tau, \mathrm{X})\,\overset{\circ}{\varepsilon}_{jl}(\mathbf{x}')\frac{(d\mathbf{x}')}{R} \tag{5.8}$$

where $E^{ik}_{ij}(\mathbf{x}, \tau)$ is a two-dimensional Green's tensor describing the field of elastic deformations $\varepsilon_{ik}(\mathbf{x}, \tau) = \varepsilon^\circ{}_{il}\, E^{ik}_{il}(\mathbf{x}, \tau)$ for an elementary source of a two-dimensional field

$$\varepsilon_{ij}{}^\circ(\mathbf{x}) = \varepsilon_{ij}{}^\circ \int\limits_{-\infty}^{\infty} \delta(\mathbf{x} - \tau s)\,\tau\, ds \tag{5.9}$$

5.3°. Green's tensor for internal distortions. If the asymmetric internal distortion tensor $u_{ij}{}^\circ$ is given, then the tensor $\sigma_{ij}{}^k$ defining the displacement fields u_k will again be the Green's tensor. Unlike the case 5.2°, we can construct not only an elastic deformation tensor (5.7), but also an asymmetric elastic distortion tensor

$$u_{ik} = u_{k,i} - \overset{\circ}{u}_{i..} = \int [\sigma^k_{mn,i}(\mathrm{X})\,\overset{\circ}{u}_{nm}(\mathbf{x}') - \delta(\mathrm{X})\,\overset{\circ}{u}_{ik}(\mathbf{x}')]\,(d\mathbf{x}') =$$

$$= \int [\sigma^k_{mn,i}(\mathrm{X})\,\overset{\circ}{u}_{nm}(\mathbf{x}') + X_j\delta_{,i}(\mathrm{X})\,\overset{\circ}{u}_{j..}(\mathbf{x}')]\,(d\mathbf{x}') =$$

$$= \int [\sigma^k_{mn,i}(\mathrm{X})\,\overset{\circ}{u}_{nm}(\mathbf{x}') + X_j\sigma^k_{mn,in}(\mathrm{X})\,\overset{\circ}{u}_{jm}(\mathbf{x}')]\,(d\mathbf{x}') = \int [\sigma^k_{mn,i}(\mathrm{X})\,X_j\overset{\circ}{u}_{jm,n}(\mathbf{x}')]\,(d\mathbf{x}')$$

5.4°. Green's tensor for dislocations. We are given a vortex source and the dislocation density tensor

$$\alpha_{ij} = -e_{ikl}u_{lj,k} = e_{ikl}\overset{\circ}{u}_{lj,k} \tag{5.10}$$

Using the fact that

$$\int [X_j\sigma^k_{mn,i}(\mathrm{X})\,\overset{\circ}{u}_{nm,j}(\mathbf{x}')]\,(d\mathbf{x}') = 0$$

which can easily be confirmed performing the integration by parts and assuming that the homogeneity index of $\sigma^k_{mn,i}$ is equal to -3, we can transform (5.9) into

$$u_{ik} = \int X_j\sigma^k_{mn,i}\,[\overset{\circ}{u}_{jm,n}(\mathbf{x}') - \overset{\circ}{u}_{nm,j}(\mathbf{x}')]\,(d\mathbf{x}') = \int \sigma^k_{mn,i}(\mathrm{X})\,e_{njl}X_j\alpha_{lm}(\mathbf{x}')\,(d\mathbf{x}') \tag{5.11}$$

which corresponds to (4.2). The elementary, two-dimensional field source (rectilinear dislocation) is then given by

$$\alpha_{ij}(\mathbf{x}) = \tau_i b_j \int\limits_{-\infty}^{\infty} \delta(\mathbf{x} - \tau s)\, ds \tag{5.12}$$

which corresponds to (4.1). In the case of elastic distortion resulting from the rectilinear dislocation where the Burgers vector $\mathbf{b}$ is given by

$$u_{ik}(\mathbf{x}, \tau, \mathbf{b}) = b_m u^m_{ik}(\mathbf{x}, \tau)$$

we find, from (5.11),

$$u^m_{i..}(\mathbf{x}, \tau) = e_{njl}x_j\tau_l \int\limits_{-\infty}^{\infty} \sigma^k_{mn,i}(\mathbf{x} - \tau s)\, ds \tag{5.13}$$

from which, using the Theorem 2, we obtain the following expression for the generalized Green's tensor:

$$e_{njl}\tau_j\sigma^k_{mn,i}(\tau) = -\frac{1}{2}\frac{\partial}{\partial x_l}\left[x_j x_n \frac{\partial^2}{\partial\tau_j\partial\tau_n} u^m_{ik}(\mathbf{x}, \tau)\right]$$

and

$$u_{ik}(\mathbf{x}) = -\frac{1}{2}\frac{\partial^2}{\partial x_l \partial x_j}\left[\delta_{lq}\tau_j + \delta_{jq}\tau_l + \tau_l\tau_j\frac{\partial}{\partial\tau_q}\right]\int u^m_{ik}(\tau, \mathbf{X})\,\alpha_{qm}(\mathbf{x}')\,(d\mathbf{x}') \tag{5.14}$$

for the distortion field resulting from the given distribution of dislocations, in terms of the distortion field (5.13) caused by rectilinear dislocations.

5.5°. Green's tensor for the deformation incompatibility. Let a deformation incompatibility tensor be given in the form of a vortex source

$$\eta_{ij} = -e_{ikm}e_{jlm}\varepsilon_{mn,kl} = e_{ikm}e_{jln}\overset{\circ}{\varepsilon}_{mn,kl} \tag{5.15}$$

Using Expression (5.7) for the field of elastic deformation generated by the given internal deformation distribution and following the procedure used in deriving the generalized Green's tensor for the dislocations, we obtain

$$\varepsilon_{ij,k}(\mathbf{x}) = \frac{1}{4}e_{mps}e_{nqt}\int X_p X_q\,[\sigma^i_{mn,jk}(\mathbf{X}) + \sigma^j_{mn,ik}(\mathbf{X})]\,\eta_{st}(\mathbf{x}')\,(d\mathbf{x}') \tag{5.16}$$

The expression

$$\eta_{ij} = \tau_i\tau_j\tau^{-1}\int_{-\infty}^{\infty}\delta(\mathbf{x} - \tau s)\,ds \tag{5.17}$$

defines the elementary, two-dimensional field source.

By Theorem 2, we have

$$\varepsilon_{ij,k}(\mathbf{x}) = -\frac{1}{8}\frac{\partial^2}{\partial\tau_s\partial\tau_t}\tau_m\tau_n\tau_l\frac{\partial^3}{\partial x_m\partial x_n\partial x_l}\int R\varepsilon_{ij,k}(\tau, \mathbf{X})\,\eta_{st}(\mathbf{x}')\,(d\mathbf{x}') \tag{5.18}$$

Differentiating (5.16) and (5.18) with respect to x_k we obtain the Laplacian of the elastic deformations, and this, in turn, can be used to construct ε_{ij} with the aid of the given Green's function of the Poisson equation.

6. The method of constructing Green's tensors discussed in Section 5 can be made more efficient and uniform if, instead of the classical Green's tensor $G_{ij}(\mathbf{x})$ of the theory of elasticity, we use as a starting point the tensor

$$G_{ijkl}(\mathbf{x}) = -\frac{1}{8\pi}G_{ijmn}\int R_{,pplm}G_{kn}(\mathbf{x}')\,(d\mathbf{x}') \tag{6.1}$$

which can be constructed [3] as easily, as G_{ij}. Since $G_{ijkl}(\alpha, \mathbf{x}) = |\alpha|^{-1}G_{ijkl}\,\mathbf{x}$ we have, by Theorem 1,

$$G_{ijkl}(\tau) = -\frac{1}{2\tau}x_m\frac{\partial}{\partial x_m}\Phi_{ijkl}(\mathbf{x}, \tau) \tag{6.2}$$

where $\Phi_{ijkl}(\mathbf{x}, \tau)$ is a two-dimensional analog of the tensor $G_{ijkl}(\mathbf{x})$. From (6.1) the following formula follows:

$$C_{ijmn}G_{kn,m} = G_{ijkl,l} \tag{6.3}$$

enabling us to express Green's tensors given in 5.2° to 5.4° in terms of the tensor G_{ijkl}

$$u_k(\mathbf{x}) = -\int G_{ijkl,l}(\mathbf{X})\,\varepsilon_{ij}{}^{\circ}(\mathbf{x}')\,(d\mathbf{x}') \tag{6.4}$$

$$\varepsilon_{ij}(\mathbf{x}) = -\frac{1}{2}\int[G_{mnij,kk} + G_{mnji,kk}]\overset{\circ}{\varepsilon}_{mn}(\mathbf{x}')\,(d\mathbf{x}') - \varepsilon_{ij}{}^{\circ} \tag{6.5}$$

$$u_{ik}(\mathbf{x}) = -\int G_{mnki,pp}e_{njl}X_j\alpha_{lm}(\mathbf{x}')\,(d\mathbf{x}') \tag{6.6}$$

and obtain an explicit form of the Green's tensors for the elastic deformation discussed in 5.5°

$$\varepsilon_{ij}(\mathbf{x}) = -\frac{1}{2}e_{mkp}\int[e_{njq}G_{mnik} + e_{niq}G_{mnjk}]\,\eta_{pq}(\mathbf{x}')\,(d\mathbf{x}') \tag{6.7}$$

as well as the following internal stresses

$$\sigma_{ij}(\mathbf{x}) = -C_{ijkl} e_{mtp} e_{neq} \int G_{mnlt} \eta_{pq}(\mathbf{x}')(d\mathbf{x}') \tag{6.8}$$

In particular we note that, using the tensor G_{ijkl} we can construct a dislocation field of a unit dislocation loop (an analog of the Burges formula for the anisotropic medium)

$$u_i = \frac{b_i}{4\pi}\Omega + e_{ljm} b_k \oint_C G_{klij}(\mathrm{X})\, dx_m' \tag{6.9}$$

7. The above results admit a simple geometric interpretation. Since action of the sources in different directions in the anisotropic medium could not be compared with each other we have made use of the similarity rule (2.1) for the sources acting along the same direction replacing the existing distribution of sources, with another distribution exerting the same action on the point under consideration. Then we can base the construction of the n-dimensional Green's function $G(\mathbf{x})$ in terms of an $(n-1)$-dimensional Green's function $\Phi(\mathbf{x}, \tau)$, on the process of replacement of a line source of constant strength corresponding to the function $\Phi(\mathbf{x}, \tau)$, by another line source situated not along the τ-axis, but along the $\mathbf{x}$-axis, and of varying strength distributed along the source according to the law: s^{k-2} sign s.

Indeed

$$\Phi(\mathrm{x}, \tau) = \int_{-\infty}^{\infty} G(\mathrm{x} - \tau s)\, \tau ds =$$

$$= \int_{-\infty}^{\infty} s^k \operatorname{sign} s G\left(\frac{x}{s} - \tau\right) \tau ds = -\int_{-\infty}^{\infty} (-s')^{k-2} \operatorname{sign} s' G(\tau - \mathrm{x}s')\, \tau\, ds' \tag{7.1}$$

where $s' = s^{-1}$, and we find that the $(k-1)$-th derivative of the field in the direction of the source, corresponds to the point source field, i.e. yields a Green's function.

The similarity rule can be used in the same manner to express elastic distortions in terms of the first moment of the dislocation distribution, and the gradient of elastic deformations in terms of the second moment of the defomation incompatibility distribution. Geometric approach to the analysis of various source fields based on the similarity relations was developed by the authors in [6], while [7] deals with the application of the method to the theory of dislocations.

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